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AUTHOR(S):

Ito, Hirotada

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D-closedness of paratangent space

近畿大学大学院総合理工学研究科 伊藤 洋忠 (Hirotada Ito)

Interdisciplinary Graduate School
for Comprehensive Science and Engineering, Kinki University

We consider a kind of higher order tangent space of a closed subset $X \subset \mathbb{R}^n$. There is an easy algebraic definition of the higher order tangent bundle $\mathcal{T}$ of X following the construction of the Zariski tangent space in algebraic geometry, which is called *the Zariski paratangent bundle* by Bierstone, Milmannn and Pawłucki [BMP1]. In 1934, Whitney posed the extension problem of finding the condition for a function on X of to be extendable to a C^d function on $\mathbb{R}^n$. As an approach to this problem, Bierstone, Milmann and Pawłucki [BMP1] have introduced *the higher order paratangent bundle* τ of a subset X of a Euclidean space, generalising Glaeser's idea. (We often omit the term "higer order".) The paratangents of order d of X form a fibred space over X. They [BMP1] and Izumi [I] used this notion to solve Whitney's problem in some restricted cases [BMP1]. We are interested in characteristing the paratangent bundle as a subset of the higher order tangent bundle of the ambient Euclidean space.

A general higer order tangent of $\mathbb{R}^n$ at a can be expressed as a linear combination of higher order derivatives of the Dirac delta function δ_a. Hence we express it by the total symbol of the linear combination, a polynomial in the dual coordinates ξ_a, The purpose of this note is to show that the paratangent space of a set X is D- closed which implies that the space is invariant with respect to partial differentiations by the components of ξ_a. We concider that this is an important property which characterises the paratangent bundle.

Recently Fefferman has given answer to the Whitney problem above and given its refinement. According to Bierstone, Milman, Pawłuki [BMP1], Fefferman's result is a solution to a variant of their conjecture [BMP2]. In this variant they use the third paratangent bundle T. It is known that $\tau \subset T \subset \mathcal{T}$. Their results implies that $\tau = T = \mathcal{T}$ ([BMP2]). It is easy to see that Zariski paratangent apace is D-closed. Hence the result $\tau = T = \mathcal{T}$ implies that they are D-closed. Since this argument requires the difficult theory of Fefferman [F], we show a direct simple proof.

1 Zariski paratangent bundle

We use the following notations

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \quad (\alpha_i \in \mathbb{N} \cup \{0\}),$$

$$e_i := (0, \dots, 0, \underset{i}{\underline{1}}, 0, \dots, 0).$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!,$$
$$x = (x_1, x_2, \dots x_n), \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

for multi exponent and multi indeces. Let X be a subset of a Euclidean space $\mathbb{R}^n$. Let $\mathcal{P}_d$ be the $\mathbb{R}$ vector space of polynomials on $\mathbb{R}^n$ of degree equal to or less than d and let $\mathcal{P}_d^*$ be its dual vector space. We define functional $\xi_{ai} \in \mathcal{P}_d^*$ by $\xi_{ai} f := \dfrac{\partial f}{\partial x_i}(a)$. We put

$$\xi_a := (\xi_{a1}, \dots, \xi_{an}),$$

Let $f \in \mathcal{P}_d$ and $\xi_a^{(p)} : \mathcal{P}_d \to \mathbb{R}$ denote the assignment

$$f \mapsto \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}(a).$$

This is an element of $\mathcal{P}_d^*$. For example,

$$\mathbf{1}_a(f) = \xi_a^{(0)}(f) := f(a), \quad \xi_a^{(e_1)}(f) = \xi_{a1} := \frac{\partial f}{\partial x_1}(a),$$

$$\xi_a^{(e_1+e_2)}(f) := \xi_{a1}\xi_{a2}(f) := \frac{\partial^2 f}{\partial x_1 \partial x_2}(a).$$

The functional $\mathbf{1}_a$ is nothing but the Dirac delta function supported at a.

In the dual space $\mathcal{P}_d^*$, any higher partial derivatives of ξ_b can be expressed as a linear combination of those of ξ_a. This is an immediate consequence of Taylor's formula.

Lemma 1.1

$$\xi_b^\alpha = \sum_{|\alpha|+|\beta| \le d} \frac{1}{\beta!}(b-a)^\beta \xi_a^{\alpha+\beta}.$$

Definition 1.2 Let V be a real vector space of dimension d ($< \infty$). A bundle (of linear subspaces of V) over X is a subset E of $X \times V$ such that, for all $\boldsymbol{a} \in X$, the fible $E_{\boldsymbol{a}} := \{v \in V \ : \ (\boldsymbol{a}, v) \in E\}$ is a linear subspace of V. We don't require local triviality.

Definition 1.3 If $V \subset \mathcal{P}_d$, let

$$V^{\perp} := \{\eta \in \mathcal{P}_d : \eta(f) = 0 \ (f \in V)$$

denote the subspace of annihilators of V in the dual space $\mathcal{P}_d^*$. Let $I^d(X) \subset C^d(\mathbf{R}^n)$ denote the ideal of C^d functions that vanish on X and $T_{\boldsymbol{a}}^d I^d(X)$ denote the set of the Taylor series of degrees d of $I^d(X)$ at $\boldsymbol{a} \in X \subset \mathbf{R}^n$. Then the Zariski paratangent bundle of order d, $\mathcal{T}^d(X)$, is the subbundle of $X \times \mathcal{P}_d^*$ with fibre $\mathcal{T}_{\boldsymbol{a}}^d = (T_{\boldsymbol{a}}^d I^d(X))^{\perp}$, for each $\boldsymbol{a} \in X$.

2 D-closedness of Zariski paratangent bundle

Definition 2.1 We call a subspace $E_{\boldsymbol{a}}$ of a fibre of a bundle $E \subset X \times V$ at $\boldsymbol{a}$ D-closed if it is invariant with respect to partial differentiations with respect to the components ξ_{ai} of $\xi_{\boldsymbol{a}}$.

In view of 1.1, the property D-closedness is independent of the choice of the point $\boldsymbol{a}$.

Theorem 2.2 *$\mathcal{T}^d(X)$ is D-closed.*

We put $\eta := \sum_{\alpha} c_\alpha \xi_{\boldsymbol{a}}^\alpha \in \mathcal{T}^d(X)$ $(c_\alpha \in \mathbb{R})$ and $g := T_{\boldsymbol{a}}^d f$. Here g is the Taylor series of degree d of C^d function f at $\boldsymbol{a} \in X \subset \mathbf{R}^n$.

We have

$$g := \sum_{|\beta| \le d} \frac{1}{\beta!}(x - \boldsymbol{a})^\beta f^{(\beta)}(\boldsymbol{a}) = \sum_{|\beta| \le d} c'_\beta (x - \boldsymbol{a})^\beta \quad \left(c'_\beta = \frac{1}{\beta!} f^{(\beta)}(\boldsymbol{a})\right),$$

$$\frac{\partial \eta}{\partial \xi_{a1}}(g) = \left(\frac{\partial}{\partial \xi_{a1}} \sum_{\alpha} c_\alpha \xi_{\boldsymbol{a}}^\alpha\right) \sum_{|\beta| \le d} c'_\beta (x - \boldsymbol{a})^\beta = \sum_{\alpha} c_\alpha \sum_{|\beta| \le d} c'_\beta \frac{\partial \xi_{\boldsymbol{a}}^\alpha}{\partial \xi_{a1}} (x - \boldsymbol{a})^\beta,$$

$$\frac{\partial \xi_a^\alpha}{\partial \xi_{a1}}(x-a)^\beta = \alpha_1 \xi_{a1}^{\alpha_1 - 1} \xi_{a2}^{\alpha_2} \cdots \xi_{an}^{\alpha_n}(x_1 - a_1)^{\beta_1}(x_2 - a_2)^{\beta_2} \cdots (x_n - a_n)^{\beta_n}$$

$$= \alpha_1(\alpha_1 - 1)!\alpha_2! \cdots \alpha_n! = \begin{cases} \alpha! & (\alpha_1 = \beta_1 + 1,\ \alpha_2 = \beta_2,\ \ldots,\ \alpha_n = \beta_n) \\ 0 & (\text{otherwise}). \end{cases}$$

Since

$$\xi_a^\alpha(x_1 - a_1)(x-a)^\beta = \xi_a^\alpha(x_1 - a_1)^{\beta_1 + 1}(x_2 - a_2)^{\beta_2} \cdots (x_n - a_n)^{\beta_n}$$

$$= \begin{cases} \alpha! & (\alpha_1 = \beta_1 + 1,\ \alpha_2 = \beta_2,\ \ldots,\ \alpha_n = \beta_n) \\ 0 & (\text{otherwise}). \end{cases}$$

We have

$$\frac{\partial \xi_a^\alpha}{\partial \xi_{a1}}(x-a)^\beta = \xi_a^\alpha(x_1 - a_1)(x-a)^\beta.$$

Therefore we have

$$\frac{\partial \eta}{\partial \xi_{a1}}(g) = \sum_\alpha c_\alpha \sum_{|\beta| \le d} c'_\beta \frac{\partial \xi_a^\alpha}{\partial \xi_{a1}}(x-a)^\beta$$

$$= \sum_\alpha c^\alpha \xi_a^\alpha \sum_{|\beta| \le d} c'_\beta (x_1 - a_1)(x-a)^\beta$$

$$= \eta((x_1 - a_1)g).$$

If $g \in I^d(X)$, we have $(x_1 - a_1)g \in I^d(X)$.
This implies

$$\frac{\partial \eta}{\partial \xi_{a1}}(g) = 0.$$

Thus $\mathcal{T}^d(X)$ is D-closed with respect to $\dfrac{\partial}{\partial \xi_{a1}}$. □

3 Paratangent space [BMP1]

Let us put

$$E_0 := \{(a,\ \lambda\xi_a) : a \in X,\ \lambda \in \mathbb{R},\ \xi_a \in \mathcal{P}_d^*\},$$

Let us fix a natural number $N \in \mathbb{N} \cup \{0\}$. We define E_k inductively as follows. If E_k is defined, we put

$$\Delta E_k := \Big\{(a_0^k,\ a_1^k, \ldots, a_N^k,\ \eta_0^k + \cdots + \eta_N^k) : a_\nu^k \in X,\ \eta_\nu^k \in E_{k,\,a_\nu},$$

$$|a_\nu^k - a_0^k|^{d-|\alpha|}|\eta_\nu^k(x - a_\nu^k)^\alpha| \leq 1,\ (|\alpha| \leq d,\ 0 \leq \nu \leq N)\Big\}$$

$$\subset X^{N+1} \times \mathcal{P}_d^*.$$

and

$$E_k' := \pi\left(\overline{\Delta E_k} \bigcap \{(a, \ldots, a, \eta) : a \in X,\ \eta \in \mathcal{P}_d^*\}\right),$$

where $\pi : X \times X \times \cdots \times X \times \mathcal{P}_d^* \longrightarrow X \times \mathcal{P}_d^*$ denotes the canonical projection onto the first factor times $\mathcal{P}_d^*$.

We put

$$E_{k+1} := \bigcup_{a \in X}(\{a\} \times \operatorname{Span} E_{k,a}') \subset X \times \mathcal{P}_d^*$$

where SpanE_k' denotes the liner span of E_k' in the fibre. The sequence $E_0 \subset E_1 \subset E_2 \subset \cdots$ stabilizes and we have

$$E_k = E_{2\dim \mathcal{P}_d^*} \quad (k \geq 2\dim \mathcal{P}_d^*)$$

as a general property of Glaeser operation [G], [BMP1]. The saturation $\tau_N^d(X) := E_{2\dim \mathcal{P}_d^*}$ is called the paratangent bundle of order d of X and the fibre $\tau_{N,\,a}^d(X)$ is called the paratangent space of order d of X at a.

Remark 3.1 *We can replace the control condition*

$$|a_\nu^k - a_0^k|^{d-|\alpha|}|\eta_\nu^k(x - a_\nu^k)^\alpha| \leq 1$$

by

$$|a_\nu^k - a_0^k|^{d-|\alpha|}|\eta_\nu^k(x - a_\nu^k)^\alpha| \leq c$$

with any $c > 0$ independent of k.

4 D-closedness of τ_N^d

We have defined that a liner subspace $A \subset \mathcal{P}_d^* = \mathbb{R}[\xi_a]$ is called D-closed if it is closed with respect to all the derivations $\dfrac{\partial}{\partial \xi_{ai}}$ $(1 \le i \le n)$. Our main result is the assertion that τ_N^d is D-closed.

Theorem 4.1 *The fibre $(\tau_N^d f)_a$ over a is D-closed.*

This theorem follows from the following lemmas.

Lemma 4.2 *Since an element of ΔE_k can be expressed as a linear combination of $\xi_{a_\nu}^\alpha$, we have*

$$\eta_\nu^k := \sum_\alpha A_\alpha^k \xi_{a_\nu^k}^\alpha \in \tau_N^d(X) \quad (A_\alpha^k \in \mathbb{R}).$$

If the control condition

$$|a_\nu^k - a_0^k|^{d-|\alpha|} \left|\eta_\nu^k(x - a_\nu^k)^\alpha\right| \le 1$$

is satisfied, we have

$$|a_\nu^k - a_0^k|^{d-|\alpha|} \left|\frac{\partial \eta_\nu^k}{\partial \xi_{a_0^k i}}(x - a_\nu^k)^\alpha\right| \le c$$

for a_ν^k in a sufficiently small neighborhood of a_0^k and for constant $c > 0$ dependent only on n and d.

We have only to prove that in the case $i = 1$ without losing generality. We have

$$\frac{\partial \eta_\nu^k}{\partial \xi_{a_0^k 1}} = \frac{\partial}{\partial \xi_{a_0^k 1}} \sum_\alpha A_\alpha^k \xi_{a_\nu^k}^\alpha = \sum_\alpha A_\alpha^k \sum_{\substack{|\alpha| + |\beta| \le d \\ \alpha + \beta - e_1 \ge 0}} \frac{1}{\beta!}(a_\nu^k - a_0^k)^\beta (\alpha_1 + \beta_1) \xi_{a_0^k}^{\alpha+\beta-e_1}$$

by Lemma 1.1. Here $\alpha \ge \beta$ means $\alpha_i \ge \beta_i$ $(1 \le \forall i \le n)$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. Accordingly we have

$$\left|\frac{\partial \eta_\nu^k}{\partial \xi_{a_0^k 1}}(x - a_\nu^k)^\alpha\right| = \left|\sum_\alpha A_\alpha^k \sum_{\substack{|\alpha| + |\beta| \le d \\ \alpha + \beta - e_1 \ge 0}} \frac{1}{\beta!}(a_\nu^k - a_0^k)^\beta (\alpha_1 + \beta_1) \xi_{a_0^k}^{\alpha+\beta-e_1}(x - a_\nu^k)^\alpha\right|$$

$$\leq \sum_{\alpha} |A^k_\alpha| \sum_{\substack{|\alpha|+|\beta|\leq d \\ \alpha+\beta-e_1\geq 0}} \frac{\alpha_1+\beta_1}{\beta!} |a^k_\nu - a^k_0|^\beta \alpha!(1+|a^k_{01}-a^k_{\nu 1}|).$$

Since

$$|a^k_\nu - a^k_0|^{d-|\alpha|}|\eta^k_\nu(x-a_\nu)^\alpha| = |a^k_\nu - a^k_0|^{d-|\alpha|}|A^k_\alpha|\alpha! \leq 1,$$

we have

$$|a^k_\nu - a^k_0|^{d-|\alpha|}\left|\frac{\partial \eta^k_\nu}{\partial \xi_{a^k_0 1}}(x-a^k_\nu)^\alpha\right| \leq \sum_\alpha \sum_\beta \frac{\alpha_1+\beta_1}{\beta!}|(a^k_\nu - a^k_0)^\beta|\{\alpha!(1+|a^k_{01}-a^k_{\nu 1}|)\}.$$

When we consider that a^k_ν is sufficiently close to a^k_0, we can assume

$$|a^k_\nu - a^k_0| \leq \frac{1}{d}\frac{(n!d!)^2}{\{(n+d)!\}^2}.$$

Hence we have

$$|a^k_\nu - a^k_0|^{d-|\alpha|}\left|\frac{\partial \eta^k_\nu}{\partial \xi_{a^k_0 1}}(x-a^k_\nu)^\alpha\right| \leq \frac{((n+d)!)^2}{(n!d!)^2} d \left\{\frac{1}{d}\frac{(n!d!)^2}{((n+d)!)^2}\right\}^d \left\{1+\frac{1}{d}\frac{(n!d!)^2}{((n+d)!)^2}\right\}.$$

Let c denotes the right constant. Then we obtain the desired estimate. □

Lemma 4.3 *If $b \to a$ and $\eta \to \eta'$, we have* $\dfrac{\partial \eta}{\partial \xi_{bi}} \longrightarrow \dfrac{\partial \eta'}{\partial \xi_{ai}}$.

Proof. We have only to prove that in the case $i = 1$ without losing generality. We know that

$$\frac{\partial \eta}{\partial \xi_{b1}} = \sum_j \frac{\partial \xi_{aj}}{\partial \xi_{b1}}\frac{\partial \eta}{\partial \xi_{aj}} = \sum_j \sum_{\substack{|\beta+e_j|\leq d \\ \beta+e_j-e_1\geq 0}} \frac{1}{\beta!}(a-b)^\beta(\beta_1+\delta_{j1})\xi_b^{\beta+e_j-e_1}\frac{\partial \eta}{\partial \xi_{aj}}$$

because of

$$\xi_{aj} = \sum_{|\beta+e_j|\leq d}\frac{1}{\beta!}(a-b)^\beta \xi_b^{\beta+e_j}.$$

We have $\sum_j \delta_{j1}\xi_a^{e_j-e_1} = 1$. Here δ_{ij} denotes the Kronecker delta ($\delta_{ii} = 1$, $\delta_{ij} = 0$ $(i \neq j)$). Therefore we have $\dfrac{\partial \eta}{\partial \xi_{b1}} \longrightarrow \dfrac{\partial \eta}{\partial \xi_{a1}}$ $(b \to a)$. Since the linear map

$\dfrac{\partial}{\partial \xi_{a1}} : \mathcal{P}_d^* \longrightarrow \mathcal{P}_d^*$ is continuous as a linear map for the vector space of the finite dimension, we have $\dfrac{\partial \eta}{\partial \xi_{a1}} \longrightarrow \dfrac{\partial \eta'}{\partial \xi_{a1}}$ $(\eta \to \eta')$ is satisfied. Thus, if $\boldsymbol{b} \to \boldsymbol{a}$ and $\eta \to \eta'$ are satisfied, we have $\dfrac{\partial \eta}{\partial \xi_{b1}} \longrightarrow \dfrac{\partial \eta'}{\partial \xi_{a1}}$. $\square$

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